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A METHOD FOR SOLVING THE FIRST INITIAL-AND-BOUNDARY-VALUE PROBLEM OF THE LINEAR THEORY OF ELASTICITY FOR ISOTROPIC BODIES†

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A quadrature of the solution of the first dynamic problem of the linear theory of elasticity in which the deformable body occupies a finite volume and is bounded by a piecewise-smooth surface, is obtained. The material of the body is assumed to be homogeneous and isotropic. It is proved that the quadrature satisfies a system of equations, as well as the initial and boundary conditions of the original problem. © 1998 Elsevier Science Ltd. All rights reserved.

We will consider the first initial-and-boundary-value problem of the dynamic theory of elasticity

$$\begin{gathered}
\sigma_{ij,j}(\mathbf{x},t)+F_i(\mathbf{x},t)=\rho\ddot{U}_i(\mathbf{x},t)\\
\sigma_{ij}(\mathbf{x},t)=\Gamma_{ijpq}\varepsilon_{pq}(\mathbf{x},t),\ \ \varepsilon_{ij}(\mathbf{x},t)=\{U_{i,j}(\mathbf{x},t)+U_{j,i}(\mathbf{x},t)\}/2\\
U_i(\mathbf{x},0)=U_{i0}(\mathbf{x}),\ \ \dot{U}_i(\mathbf{x},0)=U_{i1}(\mathbf{x});\ \ U_i(\mathbf{x}_S,t)=U_i^0(\mathbf{x}_S,t)
\end{gathered}\tag{1}$$

Here $\sigma_{ij}(\mathbf{x}, t)$, $\varepsilon_{ij}(\mathbf{x}, t)$, Γ_{ijpq} are the components of the stress and strain tensors and the tensor of elasticity constants, $F_i(\mathbf{x}, t)$, $U_j(\mathbf{x}, t)$ are the components of the vector of mass force and the displacement vector, $U_{i0}(\mathbf{x})$ $U_{i1}(\mathbf{x})$ is the initial distribution of the displacements and rates of displacements in the body, which is bounded by the surface S; $U^0{}_i(\mathbf{x}_S, t)$ is the boundary value of the displacement vector and $\mathbf{x}$ and t are the space coordinates and the actual time. In the case under consideration

$$\Gamma_{ijpq}=\lambda\delta_{ij}\delta_{pq}+\mu(\delta_{ip}\delta_{jq}+\delta_{iq}\delta_{jp})$$

where λ and μ are the Lamé constants.

We apply a Laplace transformation with respect to time and a multiple Fourier transformation with respect to the coordinates. Applying a Laplace transformation to the system of equations (1), we obtain

$$\begin{gathered}
\sigma^*_{ij,j}(\mathbf{x},p)+\Phi^*_i(\mathbf{x},p)=\rho p^2U^*_i(\mathbf{x},p)\\
\Phi^*_i(\mathbf{x},p)=F^*_i(\mathbf{x},p)+\rho p^2U_{i0}(\mathbf{x})+\rho U_{i1}(\mathbf{x})\\
\sigma^*_{ij}(\mathbf{x},p)=\Gamma_{ijpq}\varepsilon^*_{pq}(\mathbf{x},p),\ \ \varepsilon^*_{ij}(\mathbf{x},p)=\{U^*_{i,j}(\mathbf{x},p)+U^*_{j,i}(\mathbf{x},p)\}/2\\
U^*_i(\mathbf{x}_S,p)=U^{0*}_i(\mathbf{x}_S,p)
\end{gathered}\tag{2}$$

The asterisk denotes the Laplace transform and p is the transformation parameter.

We will solve boundary-value problem (2) by the Fourier transformation method using potential theory and a fundamental solution. We place the body, which has a volume V and a bounded surface S, in a larger volume V_1 in such a way that the boundary surface S_1 has no points in common with the surface S. The set of points belonging to V_1 and not belonging to V forms the volume V_2 bounded by the surfaces S and S_1. Since the equation

$$\sigma^*_{ij,j}(\mathbf{x},p)=\rho p^2U^*_i(\mathbf{x},p)$$

has a fundamental solution (the Kupradze matrix [1])

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$$R_{kj}(\mathbf{x},p)=\sum_{l=1}^{2}\left(\delta_{kj}\alpha_l+\beta_l\frac{\partial^2}{\partial x_k \partial x_j}\right)\frac{e^{ik_l|\mathbf{x}|}}{|\mathbf{x}|}$$

$$\alpha_l=\delta_{2l}(2\pi\mu)^{-1};\quad k_l^2=k_1^2=\rho p^2(\lambda+2\mu)^{-1}\quad \text{for}\quad l=1$$

$$\beta_l=(-1)^l(2\pi\rho p^2)^{-1};\quad k_l^2=k_2^2=\rho p^2\mu^{-1}\quad \text{for}\quad l=2$$

the particular solution of the first equation of (2) can be written in the form

$$U_k^*(\mathbf{x},p)=\int_{V_1} R_{kj}(\mathbf{x}-\mathbf{y},p)\Phi_j^{1*}(\mathbf{y},p)d\mathbf{y} \tag{3}$$

Let the function $\Phi_j^{1*}(\mathbf{y},p)$ be the same as $\Phi_i^*(\mathbf{x},p)$ in V. We obtain

$$U_k^*(\mathbf{x},p)=\int_{V} R_{kj}(\mathbf{x}-\mathbf{y},p)\Phi_j^{*}(\mathbf{y},p)d\mathbf{y}+\int_{V_2} R_{kj}(\mathbf{x}-\mathbf{y},p)\Phi_j^{2*}(\mathbf{y},p)d\mathbf{y} \tag{4}$$

Here $\Phi_j^{2*}(\mathbf{y},p)$ is the Laplace transform of unknown mass forces distributed in the volume. We define them in such a way that the transform $U_k^*(\mathbf{x},p)$ satisfies the boundary conditions of problem (2) on the surface S. Then in V expression (3) will be the required solution of problem (2). For this purpose, we put $\mathbf{x}$ on the surface S in (3), multiply (3) by the expression

$$(n_1(\mathbf{x}_S)+n_2(\mathbf{x}_S)+n_3(\mathbf{x}_S))e^{-i\mathbf{k}\cdot\mathbf{x}_S}$$

and integrate over S. We obtain

$$\int_S n(\mathbf{x}_S)e^{-i\mathbf{k}\cdot\mathbf{x}_S}U_k^{0*}(\mathbf{x}_S,p)ds=\int_S\int_V n(\mathbf{x}_S)e^{-i\mathbf{k}\cdot\mathbf{x}_S}R_{kj}(\mathbf{x}_S-\mathbf{y},p)\Phi_j^*(\mathbf{y},p)d\mathbf{y}ds+$$
$$+\int_S\int_{V_2} n(\mathbf{x}_S)e^{-i\mathbf{k}\cdot\mathbf{x}_S}R_{kj}(\mathbf{x}_S-\mathbf{y},p)\Phi_j^{2*}(\mathbf{y},p)d\mathbf{y}ds \tag{5}$$

$$n(\mathbf{x}_S)=n_1(\mathbf{x}_S)+n_2(\mathbf{x}_S)+n_3(\mathbf{x}_S)$$

Here $n_j(\mathbf{x}_S)$ is the corresponding coordinate of the normal to S. The unknown mass forces $\Phi_j^{2*}(\mathbf{y},p)$ are found from Eq. (5).

Using the theorem on convolution and the Gauss–Ostrogradskii theorem, from (5) we obtain

$$U_k^{0**}(\mathbf{k},p)=R_{kj}^{1*}(\mathbf{k},p)\Phi_j^{**}(\mathbf{k},p)+R_{kj}^{2*}(\mathbf{k},p)\Phi_j^{2**}(\mathbf{k},p) \tag{6}$$

Here

$$U_k^{0**}(\mathbf{k},p)=\int_S n(\mathbf{x}_S)e^{-i\mathbf{k}\cdot\mathbf{x}_S}U_k^{0*}(\mathbf{x}_S,p)ds$$

$$\Phi_j^{**}(\mathbf{k},p)=\int_V e^{-i\mathbf{k}\mathbf{y}}\Phi_j^*(\mathbf{y},p)d\mathbf{y},\quad \Phi_j^{2**}(\mathbf{k},p)=\int_{V_2} e^{-i\mathbf{k}\cdot\mathbf{x}_S}\Phi_j^{2*}(\mathbf{y},p)d\mathbf{y}$$

$$R_{kj}^{r*}(\mathbf{k},p)=\int_{V_{zr}}\left[\sum_{n=1}^{3}-ik_n+\frac{\partial}{\partial x_n}R_{kj}(\mathbf{z},p)\right]e^{-i\mathbf{k}\cdot\mathbf{z}}d\mathbf{z},\quad r=1,2$$

The regions of integration V_{1z} and V_{2z} in the last equation are determined by the volumes V, V_1, V_2 and the equation $\mathbf{z}=\mathbf{x}-\mathbf{y}$.

Relations (6) form a system of three equations from which to find the three unknown integral tranforms of the mass forces $\Phi_j^{2**}(\mathbf{k},p)$. Since, as we know [1], problem (1) has a unique solution, the determinant of system (6) is not identically zero and the system has a unique solution.

Let the matrix $\mathrm{Res}_{jm}(\mathbf{k},p)$ be the inverse of $R_{kj}^{2**}(\mathbf{k},p)$, that is,

$$R_{kj}^{2*}(\mathbf{k},p)\,\mathrm{Res}_{jm}(\mathbf{k},p)=\delta_{km}$$

Then from (6) we have the equation

$$\Phi_j^{2**}(\mathbf{k},p)=\mathrm{Res}_{jm}(\mathbf{k},p)U_m^{0**}(\mathbf{k},p)-\mathrm{Res}_{jm}(\mathbf{k},p)R_{me}^{1*}(\mathbf{k},p)\Phi_e^{**}(\mathbf{k},p)$$

from which we obtain a formula for $\mathbf{U}(\mathbf{x},t)$, the solution of the original problem (1)

$$U_l(\mathbf{x},t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{pt}\Bigg\{\int_V R_{ej}(\mathbf{x}-\mathbf{y})[F_j^*(\mathbf{y},p)+\rho pU_{j0}(\mathbf{y})+\rho U_{j1}(\mathbf{y})]d\mathbf{y}+$$
$$+\int_{V_2}\mathrm{R}_{lj}(\mathbf{x}-\mathbf{y},p)\Bigg[\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}e^{i\mathbf{k}\cdot\mathbf{x}}(\mathrm{Res}_{jm}(\mathbf{k},p)U_m^{0**}(\mathbf{k},p)-$$
$$-\mathrm{Res}_{jm}(\mathbf{k},p)R_{me}^{1*}(\mathbf{k},p)\Phi_e^{**}(\mathbf{k},p))d\mathbf{k}\Bigg]\Bigg\}d\mathbf{y}dp \tag{7}$$

We will prove that the quadrature (7) is a solution of the original problem (1), that is, that (7) satisfies the original system of equations as well as the initial and boundary conditions of problem (1).

Theorem 1. The quadrature (7) satisfies the equation of motion of the initial-and-boundary-value problem (1).

Proof. We write Eq. (7) in the form

$$U_l(\mathbf{x},p)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{pt}U_l^*(\mathbf{x},p)dp \tag{8}$$

Then according to the properties of the Laplace transformation we have

$$\ddot{U}_l(\mathbf{x},t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{pt}[p^2U_l^*(\mathbf{x},p)-pU_{l0}(\mathbf{x})-U_{l1}(\mathbf{x})]dp \tag{9}$$

Using expression (3), for $\mathbf{U}_l(\mathbf{x},t)$ from (8) we have

$$U_l(\mathbf{x},t)=\int_0^t\int_{V_1}R_{lj}^1(\mathbf{x}-\mathbf{y},t-\tau)\Phi_j^1(\mathbf{y},\tau)d\mathbf{y}d\tau \tag{10}$$

Here

$$R_{lj}^1(\mathbf{x}-\mathbf{y},t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{pt}R_{lj}(\mathbf{x}-\mathbf{y},p)dp,\quad \Phi_j^1(\mathbf{y},\tau)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{pt}\Phi_j^{1*}(\mathbf{y},p)dp$$

We substitute expressions (9) and (10) into the equation of motion of problem (1). Changing to integral transforms and using (3), we obtain

$$\int_{V_1}\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{pt}K_{lj}(\mathbf{x}-\mathbf{y},p)\Phi_j^{1*}(\mathbf{y},p)d\mathbf{y}dp+\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{pt}\Phi_l^*(\mathbf{x},p)dp=0 \tag{11}$$
$$K_{lj}(\mathbf{x}-\mathbf{y},p)=L_xR_{lj}(\mathbf{x}-\mathbf{y},p)+\rho p^2R_{lj}(\mathbf{x}-\mathbf{y},p)$$

where L_x is the Lamé operator acting on the variables $\mathbf{x}$.
Here we have taken into account the relation

$$F_l(\mathbf{x},t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{pt}[\Phi_l^*(\mathbf{x},p)-\rho pU_{l0}(\mathbf{x})-\rho U_{l1}(\mathbf{x})]dp \tag{12}$$

In relation (11), according to the property of the fundamental solution, $K_{lj}(\mathbf{x}-\mathbf{y},p)$ is the kernel of the identity integral transformation and thus

$$\int_{V_1}K_{lj}(\mathbf{x}-\mathbf{y},p)\Phi_j^{1*}(\mathbf{y},p)d\mathbf{y}=\Phi_l^{1*}(\mathbf{x},p)$$

Using this relation, (11) becomes an identity, which proves the theorem.

Theorem 2. The quadrature (7) satisfies the initial conditions of problem (1).

Proof. We use the properties of the Laplace transformation. Then, from Eq. (8), we have

$$\ddot{U}_l(\mathbf{x},t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{pt} p^2 \left[U_l^*(\mathbf{x},p) - \frac{U_{l0}(\mathbf{x})}{p} - \frac{U_{l1}(\mathbf{x})}{p^2} \right] dp \tag{13}$$

Here $U_l^*(\mathbf{x},p)$ is the solution of boundary-value problem (2) and is a known function, that is, formula (13) uniquely defines $\ddot{U}_l(\mathbf{x}, t)$ as a known function. Denote it by $f(\mathbf{x}, t)$. Then by solving Eq. (13) for $U_l^*(\mathbf{x},p)$, we obtain a form of the quadrature (7) in which the initial values $U_{l0}(\mathbf{x})$ and $U_{l1}(\mathbf{x})$ are explicit

$$U_l(\mathbf{x},t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{pt} U_l^*(\mathbf{x},p)dp = \int_0^t f(\mathbf{x},t)(t-\tau)d\tau + U_{l0}(\mathbf{x}) + U_{l1}(\mathbf{x})t \tag{14}$$

Putting $t = 0$ in (14) we obtain one of the initial conditions, and differentiating (14) with respect to t and putting $t = 0$ we obtain a different initial condition for problem (1).

Theorem 3. The quadrature (7) satisfies the boundary conditions of problem (1).

Proof. We use relations (8) and (3). Then

$$U_l^*(\mathbf{x},p) = \int_{V_1} R_{lj}(\mathbf{x}-\mathbf{y})\Phi_j^{l*}(\mathbf{y},p)d\mathbf{y} \tag{15}$$

In (15) we assume that $\mathbf{x}$ is on the surface S. Multiplying Eq. (15) by

$$(n_1(\mathbf{x}_S) + n_2(\mathbf{x}_S) + n_3(\mathbf{x}_S))e^{-i\mathbf{k}\mathbf{x}_S}$$

and integrating over S, from Eq. (5) we obtain

$$\int_S n(\mathbf{x}_S)e^{-i\mathbf{k}\mathbf{x}_S}[U_l^*(\mathbf{x}_S,p) - U_{l0}^*(\mathbf{x}_S,p)]dS = 0 \tag{16}$$

Since S was an arbitrary surface, the expression in square brackets vanishes, which proves the theorem.

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